INFINITE SERIES

(B.Sc.-II, Paper-III)

Group- B

Contents: - Infinite Series,
Cauchy's general principle
of convergence for a series,
Necessary condition for convergence.

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Infinite Series

If $\{a_n\}$ be a sequence of real numbers then

$$\sum_{n=1}^{\infty} a_n \equiv a_1 + a_2 + \cdots\cdots + a_n + \cdots\cdots$$

is called an infinite series.

Let $S_n = a_1 + a_2 + \cdots\cdots + a_n$ be the nth partial sum.

Then $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3, \cdots\cdots$

$\cdots\cdots$, $S_n = a_1 + \cdots + a_n$, $\cdots\cdots$

If the sequence $\{S_n\}$ is convergent then the series $\sum_{n=1}^{\infty} a_n$ is said to be convergent series.

If the sequence $\{S_n\}$ is divergent then the series $\sum_{n=1}^{\infty} a_n$ is said to be divergent series.

If the sequence $\{S_n\}$ is an oscillating sequence the ~~the~~ series $\sum_{n=1}^{\infty} a_n$ is said to be an oscillating sequence.

Remark:→ A positive term series is either convergent or divergent.

Example:→ Show that the series

$$1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\cdots + \left(\frac{3}{4}\right)^n + \cdots\cdots \text{ converges.}$$

solution :→

Let $S_n = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-1}$

Then $S_n = \dfrac{1 - \left(\frac{3}{4}\right)^n}{1 - \frac{3}{4}} = 4\left\{1 - \left(\frac{3}{4}\right)^n\right\}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} 4\left\{1 - \left(\frac{3}{4}\right)^n\right\}$

$= 4$ ~~Ans~~.

$\therefore$ The given infinite series is convergent.

Example② :→ show that the series $1+2+3+\cdots\cdots$ diverges

solution :→

Let $S_n = 1 + 2 + \cdots\cdots + n$

Then $S_n = \frac{n(n+1)}{2}$

And $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$.

Hence the series is divergent.

Example③ :→ Show that the series $1 - 1 + 1 - 1 + \cdots\cdots + (-1)^{n+1} + \cdots$ is oscillating.

solution :→

Here $S_n = \begin{cases} 0, & \text{if } n \text{ is even.} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$

Hence the sequence $\{S_n\}$ oscillates.

Therefor, the series is oscillating.

THEOREM :→ The geometric series

$a + ar + ar^2 + \cdots\cdots + ar^{n-1} + \cdots\cdots \infty$

is (i) convergent when $-1 < r < 1$ and its sum is then $\frac{a}{1-r}$;

(ii) divergent if $r \geq 1$.

(iii) Oscillating if $r \leq -1$.

Proof:→

$\because$ The geometric series is

$a + ar + ar^2 + \cdots\cdots\cdots + ar^{n-1} + \cdots\cdots$ to ∞.

Let S_n be its partial sum.

$\therefore S_n = a + ar + \cdots\cdots + ar^n$

$$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}, \quad (r \neq 1) \longleftarrow (1)$$

case-1 :→ When $-1 < r < 1$

For such r, $\lim_{n\to\infty} r^n = 0$.

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \text{ (finite)}$$

$\therefore$ The series is convergent.

Case 2 :→ When $r = 1$

Then $S_n = a + a + \cdots + a$ (n-times)

$\therefore S_n = na$

$\therefore \lim_{n\to\infty} S_n = +\infty$ or $-\infty$ according as $a>0$ or $a<0$.

$\therefore$ The series is <u>divergent</u>.

<u>Case 3:→</u> <u>When $r=-1$.</u>

Then the series $a-a+a-a+ \cdots\cdots\cdots$.

And $S_{2n}=0$, $S_{2n-1}=a$.

$\therefore \{S_n\}$ is oscillating.

$\therefore$ The series oscillates finitely.

<u>Case 4:→</u> <u>When $r>1$.</u>

$$\because S_n = \frac{a(1-r^n)}{1-r}, \quad r\neq 1 \quad (\text{from } ①)$$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$\to +\infty \text{ if } a>0$$

$$\text{and} \to -\infty \text{ if } a<0$$

$\therefore$ The series is <u>divergent</u>.

<u>Case 5:→</u> <u>When $r<-1$.</u>

Let $r=-R$ where $R>1$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1+R}\left\{1-(-R)^n\right\}$$

$$= \frac{a}{1+R}\left\{1+(-1)^{n+1}R^n\right\}$$

$\therefore \{S_n\}$ <u>oscillates</u> <u>infinitely</u> and so the series <u>oscillates infinitely</u>.

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Infinite series

THEOREM (Cauchy's general principle of convergence for a series)

Statement :→ An infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers is convergent iff the sequence $\{S_n\}$ is a cauchy sequence, where

$S_n = a_1 + \cdots\cdots + a_n$.

That is if for any $\epsilon > 0$, $\exists\ N(\epsilon)$ s.t.

$$|S_n - S_m| < \epsilon \quad \text{for all } m, n \geq N.$$

i.e; $|a_{m+1} + a_{m+2} + \cdots\cdots + a_n| < \epsilon$, $\forall\ m, n \geq N$.

Proof :→

Lemma :→ The sequence $\{a_n\}$ is convergent iff $\{a_n\}$ is a cauchy sequence.

Proof :→ Let $\{a_n\}$ be a convergent sequence converging to the limit l.

$\therefore$ For any $\epsilon > 0$, $\exists\ N(\epsilon)$ such that

$$|a_n - l| < \frac{\epsilon}{2}, \ \forall\ n \geq N$$

and $|a_m - l| < \frac{\epsilon}{2}, \ \forall\ n \geq N$

$$\therefore |a_n - a_m| = |a_n - l + l - a_m|$$

$$\leq |a_n - l| + |a_m - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}, \ \forall\ m, n \geq N$$

$\therefore |a_n - a_m| < \epsilon$, $\forall\ m, n \geq N$

$\therefore \{a_n\}$ is a cauchy sequence.

Conversely :- Let $\{a_n\}$ be a cauchy sequence.

$\therefore$ For any $\epsilon > 0$, $\exists$ $N(\epsilon)$ such that

$\therefore |a_n - a_m| < \frac{\epsilon}{2}, \; \forall \; m,n \geq N$

$\therefore |a_n - a_N| < \frac{\epsilon}{2}$, for all $n \geq N$

$$\therefore \quad a_N - \frac{\epsilon}{2} < a_n < a_N + \frac{\epsilon}{2}, \; \forall \; n \geq N$$

Let $S = \{x \in \mathbb{R} \mid x < a_n$ for infinite number of members of $a_n\}$

$$\because \quad (a_N - \frac{\epsilon}{2}) \in S \quad \Rightarrow S \neq \phi.$$

Also $a_N + \frac{\epsilon}{2}$ is an upper bound of S.

$\therefore$ By the axiom of l.u.b. of real numbers (A non-empty set which is bounded above has a l.u.b)

$\therefore$ S has a l.u.b.

Let l.u.b. of $S = l$

$$l \in \left[a_N - \frac{\epsilon}{2}, \; a_N + \frac{\epsilon}{2}\right]$$

$$\therefore \quad |a_N - l| < \frac{\epsilon}{2}$$

$$\therefore |a_n - l| = |a_n - a_N + a_N - l|$$
$$\leq |a_n - a_N| + |a_N - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \; \forall \; n \geq N$$

$\therefore |a_n - l| < \epsilon, \quad \forall\ n \geq N.$

$\therefore \{a_n\}$ is convergent to l.

<u>Proof of Main theorem</u> :→

By lemma, $\{S_n\}$ is convergent iff $\{S_n\}$ is a cauchy sequence.

i,e; if

$$|S_n - S_m| < \epsilon, \quad \forall\ m, n \geq N(\epsilon)$$

$$\therefore |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon, \quad \forall\ m, n \geq N$$

proved.

<u>THEOREM</u> (A necessury condition for convergence)

Prove that if the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

<u>Proof</u>:→

The series is

$$a_1 + a_2 + \cdots\cdots + a_n + \cdots\cdots$$

$\because \sum a_n$ is convergent

$\therefore \lim_{n\to\infty} S_n = l$ (say)

$\because S_{n-1} = a_1 + \cdots + a_{n-1}$

$\therefore S_n - S_{n-1} = a_n$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = l - l = 0$

$\therefore \lim_{n\to\infty} a_n = 0$ proved.

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Remark:→ Example to show that the converse is not true.

Example:→ We consider the ~~~~ series (Harmonic series)

$$\sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

$$\therefore S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

If the series is convergent then by cauchy's principle of convergence for $\epsilon = \frac{1}{2}$, there exists a natural number $N(\epsilon)$ s.t.

$$\left| \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{n} \right| < \frac{1}{2} \text{ for } n > N$$

Let $n = 2N$

$$\therefore \left| \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N} \right| < \frac{1}{2} \quad \text{— (A)}$$

But,

$$\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N} > \frac{1}{2N} + \frac{1}{2N} + \cdots + \frac{1}{2N} \text{ (N times)}$$

$$= \frac{N}{2N} = \frac{1}{2}$$

$$\therefore \left| \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N} \right| > \frac{1}{2} \quad \text{— (B)}$$

(B) contradicts (A)

Hence the series is not <u>convergent</u>.

But $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$

Hence the condition is not sufficient for <u>convergence of series</u>.

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Example:→ Proved that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n+1}$ is not convergent.

Proof:→ Here $a_n = \frac{n}{n+1}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$\therefore \lim_{n\to\infty} a_n \neq 0.$$

$\therefore$ The series is not convergent & hence divergent (as it is a positive term series)

Example :- Proved that $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$ is not convergent.

Solution :→ $\because a_n = n^{\frac{1}{n}}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$\because \lim_{n\to\infty} a_n = 1 \neq 0$$

$\therefore$ The series is <u>not convergent.</u>

Theorem (~~Pringsheim~~ Pringsheim's theorem):-

Statement :→ If $\sum a_n$ be a convergent series of monotonic decreasing positive terms, then $\lim_{n\to\infty} n a_n = 0$

Proof:→

Let $S_n = a_1 + a_2 + \cdots\cdots + a_n$

$$\therefore S_{2n} = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_{2n}.$$

$\therefore S_{2n} - S_n = a_{n+1} + a_{n+2} + \cdots + a_{2n}$.

$\because$ The series Σa_n is convergent.

$\therefore \lim_{n\to\infty} S_n = l$ (finite)

and also $\lim_{n\to\infty} S_{2n} = l$

$$\therefore \lim_{n\to\infty} (a_{n+1} + a_{n+2} + \cdots + a_{2n}) = \lim_{n\to\infty} (S_{2n} - S_n)$$

$$= l - l = 0 \quad — ①$$

$\because \{a_n\}$ is decreasing.

$\therefore a_{2n} \leq a_{2n-1} \leq \cdots \leq a_{n+1}$

$\therefore 0 < n a_{2n} \leq a_{n+1} + a_{n+2} + \cdots + a_{2n}$

$\therefore \lim_{n\to\infty} 0 < \lim_{n\to\infty} n a_{2n} \leq \lim_{n\to\infty} (a_{n+1} + \cdots + a_{2n}) = 0$

$$\therefore \lim_{n\to\infty} n a_{2n} = 0 \Rightarrow \lim_{n\to\infty} 2n a_{2n} = 0 \quad — ②$$

Again,

$$0 < (2n+1) u_{2n+1} \leq (2n+1) u_{2n} = \left(\frac{2n+1}{2n}\right)(2n u_{2n})$$

$$\to 0 \text{ as } n \to \infty$$

$$\therefore \lim_{n\to\infty} (2n+1) u_{2n+1} = 0 \quad —— ③$$

from ② & ③, we have

$$\lim_{n\to\infty} n u_n = 0$$

proved.

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Example :→ Use pringsheim's theorem to prove that $\sum \frac{1}{n}$ is divergent.

Solution :→

$\because \sum \frac{1}{n}$ is a positive term series.

$\because$ The sequence $\{\frac{1}{n}\}$ is monotonically decreasing.

$\therefore$ By Pringsheim's necessary condition for convergence is that $na_n \to 0$

Hence $\lim_{n\to\infty} n \cdot \frac{1}{n} = 1 \neq 0$.

Hence the series $\sum \frac{1}{n}$ is divergent.

Thank you